Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

EXISTENCE THEOREM FOR THE NON-SELF-ADJOINT LINEAR SYSTEM OF THE SECOND ORDER.*

By H. J. Ettlinger.

We shall consider the self-adjoint linear differential equation of the second order,

$$\frac{d}{dx}\left[K(x,\lambda)\frac{du}{dx}\right] - G(x,\lambda)u = 0 \tag{1}$$

together with two linear boundary conditions,

$$U_i = A_{i1}u(a) - A_{i2}K(a)u_x(a) - A_{i3}u(b) + A_{i4}K(b)u_x(b) = 0 \qquad (i = 1, 2) \tag{2}$$

satisfying the conditions:

I. $K(x, \lambda)$ and $G(x, \lambda)$ are continuous, real functions of x in X, $(a \leqq x \leqq b)$, and for all real values of λ in the interval $\Lambda(\Lambda_1 < \lambda < \Lambda_2)$.†

II. $K(x, \lambda)$ is positive everywhere in (X, Λ).

III. The sets of real constants A_{1j} and A_{2j} are not proportional.

IV. For each value of x in X, K and G decrease (or do not increase) as λ increases.

V.‡
$$\lim_{\lambda\to\Lambda_1} - \frac{\min G}{\min K} = -\infty,$$

$$\lim_{\lambda\to\Lambda_2} - \frac{\max G}{\max K} = +\infty.$$

We seek to determine the conditions for the existence of solutions, not vanishing identically, of the system (1), (2). A value of λ, $\lambda = \bar{\lambda}$, for which the system has such a solution is called a *characteristic number*. The problem under investigation is then the following—do there exist characteristic numbers for the system (1), (2)?

We write the array of coefficients of (2) as

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \end{pmatrix} \tag{3}$$

and let D_{ij} be the determinant formed by the ith and jth columns. We note first that not all the determinants D_{ij} can vanish, since by condition

---

* Presented to the Amer. Math. Soc., Dec. 29, 1917.

† In particular Λ may be $-\infty < \lambda < +\infty$.

‡ This condition may be replaced by other sets of conditions. See Bôcher, Leçons sur les Methodes de Sturm, Chap. III, paragraphs 13–15.

III U_1 and U_2 are linearly independent. Moreover, $D_{ij} = -D_{ji}$[*] and the six determinants D_{ij} satisfy the well-known identity:

$$D_{12}D_{34} + D_{23}D_{14} + D_{42}D_{13} = 0. \tag{4}$$

Let the adjoint boundary conditions[†] of (2) be

$$V_i = B_{i1}v(a) - B_{i2}K(a)v_x(a) - B_{i3}v(b) + B_{i4}K(b)v_x(b) = 0 \qquad (i = 1, 2). \tag{5}$$

To evaluate V_1 and V_2 we choose U_3 and U_4 defined as in (2), such that U_1, U_2, U_3, and U_4 are linearly independent, or

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix} \neq 0. \tag{6}$$

Green's Theorem gives

$$[K(vu_x - uv_x)]_{x=a}^{x=b} = \sum_{i=1}^{i=4} U_i V_{5-i}. \tag{7}$$

Equating coefficients of corresponding terms in (7), i.e., of $u(a)$, $K(a)u_x(a)$, $u(b)$, $K(b)u_x(b)$ we have

$$\begin{aligned} A_{11}V_4 + A_{21}V_3 + A_{31}V_2 + A_{41}V_1 &= K(a)v_x(a), \\ A_{12}V_4 + A_{22}V_3 + A_{32}V_2 + A_{42}V_1 &= v(a), \\ A_{13}V_4 + A_{23}V_3 + A_{33}V_2 + A_{43}V_1 &= K(b)v_x(b), \\ A_{14}V_4 + A_{24}V_3 + A_{34}V_2 + A_{44}V_1 &= v(b). \end{aligned} \tag{8}$$

Cramer's method yields the solution

$$V_i = \frac{\bar{\Delta}_{5-i}}{\bar{\Delta}} \qquad (i = 1, 2, 3, 4), \tag{9}$$

where $\bar{\Delta}_{5-i}$ is the determinant obtained by replacing the $(5 - i)$th column of $\bar{\Delta}$, the conjugate of Δ, by the right-hand terms of equations (8).

We shall use the following:

Lemma. D_{ij} *is a relative invariant of weight* 1 *of a linear transformation of the coefficients* A_{1j}, A_{2j}.

Proof: Let

$$\begin{aligned} A'_{1k} &= c_1A_{1k} + c_2A_{2k}, \\ A'_{2k} &= d_1A_{1k} + d_2A_{2k}, \end{aligned} \qquad (k = 1, 2, 3, 4)$$

where

---

[*] $i = 1, 2, 3, 4$; $j = 1, 2, 3, 4$. In particular $D_{ii} = 0$.

[†] v is a solution, not vanishing identically, of equation (1). Bôcher, Leçons sur les Methodes de Sturm (1917), p. 28 ff.

$$\delta = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \neq 0.$$

This transformation represents the result of two independent linear combinations of U_1 and U_2. Let D'_{ij} be the determinant formed by the ith and jth columns of the transformed coefficients. Then

$$D'_{ij} = \begin{vmatrix} c_1A_{1i} + c_2A_{2i} & c_1A_{1j} + c_2A_{2j} \\ d_1A_{1i} + d_2A_{2i} & d_1A_{1j} + d_2A_{2j} \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} \cdot \begin{vmatrix} A_{1i} & A_{1j} \\ A_{2i} & A_{2j} \end{vmatrix} = \delta D_{ij}.$$

It follows immediately from this lemma that if any determinant of (2) vanishes, the corresponding primed determinant will vanish, and if two determinants do not vanish, the sign of their product is an invariant of the transformation. We shall include all boundary conditions of the form (2) under two cases: (1) $D_{12} \cdot D_{34} = 0$, (2) $D_{12} \cdot D_{34} \neq 0$.

Case 1. $D_{12} \cdot D_{34} = 0$.

(*a*) If $D_{12} = 0$, $D_{34} = 0$, then not all the other determinants can vanish. The coefficients of (2) will then be transformed* into

$$\begin{pmatrix} D_{13} & D_{23} & 0 & 0 \\ 0 & 0 & D_{13} & D_{14} \end{pmatrix} \tag{10}$$

and the boundary conditions (2) are Sturmian.†

(*b*) If $D_{12} \neq 0$, $D_{34} = 0$, we can choose $U_3 = u(b)$ and $U_4 = K(b)u_x(b)$. Then equations (8) yield for the coefficients of (5),

$$\begin{pmatrix} D_{14} & D_{24} & D_{12} & 0 \\ D_{13} & D_{23} & 0 & D_{21} \end{pmatrix}. \tag{11}$$

(*c*) If $D_{12} = 0$, $D_{34} \neq 0$, we can choose $U_3 = K(a)u_x(a)$ and $U_4 = u(a)$. Then we obtain from (8) for the coefficients of (5),

$$\begin{pmatrix} D_{43} & 0 & D_{23} & D_{24} \\ 0 & D_{34} & D_{13} & D_{14} \end{pmatrix}. \tag{12}$$

Let $\boldsymbol{D}_{ij}$ be the determinant of the ith and jth columns of the derived adjoint coefficients. It is obvious in *Case* 1. (*a*) that $D_{12} \cdot D_{34} = 0$, since the system is Sturmian. By inspection of (11) and (12) and applying (4), we see that for *Case* 1. (*b*) and (*c*) $\boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34} = 0$. We note, in addition, that the adjoint of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ is $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Hence, by applying the lemma, we have in all cases

---

* $\delta = D_{13} \neq 0$. If $D_{13} = 0$, we may take $D_{24} \neq 0$. If $D_{13} = D_{24} = 0$, then $D_{14} \neq 0$, etc.

† Bôcher, Leçons, p. 60.

THEOREM I. *If for the system* (1), (2) *satisfying conditions I–III,* $D_{12} \cdot D_{34} = 0$, *then for the adjoint system* (1), (5) $\boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34} = 0$; *furthermore if* $D_{12} = 0$, $D_{34} \neq 0$, *then* $\boldsymbol{D}_{12} \neq 0$, $\boldsymbol{D}_{34} = 0$, *and if* $D_{12} \neq 0$, $D_{34} = 0$, *then* $\boldsymbol{D}_{12} = 0$, $\boldsymbol{D}_{34} \neq 0$.

We now consider

Case 2. $D_{12} \cdot D_{34} \neq 0$.

(*a*) If $D_{12} = D_{34}$, the system (1), (2) is self-adjoint, and $D_{12} = \boldsymbol{D}_{12}$, $D_{34} = \boldsymbol{D}_{34}$ or $D_{12} \cdot D_{34} = \boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34}$.

(*b*) If $D_{12} \neq D_{34}$, we shall transform the coefficients of (2) by $\delta = D_{34} \neq 0$ into

$$\begin{pmatrix} D_{14} & D_{24} & D_{34} & 0 \\ D_{13} & D_{23} & 0 & D_{43} \end{pmatrix}. \tag{13}$$

Choose $U_3 = u(b)$ and $U_4 = K(b)u_x(b)$. Then for the coefficients of (5) equations (8) will yield

$$\begin{pmatrix} D_{14} & D_{24} & D_{12} & 0 \\ D_{13} & D_{23} & 0 & D_{21} \end{pmatrix}. \tag{14}$$

By use of (4) we obtain from (14)

$$\boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34} = D_{12}^3 \cdot D_{34}. \tag{15}$$

From (15) we infer immediately that the sign of $D_{12} \cdot D_{34}$ is the same as that of $\boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34}$. Hence, by applying the lemma, we have

THEOREM II. *If for the system* (1), (2) *satisfying conditions I–III,* $D_{12} \cdot D_{34} \neq 0$, *then the sign of* $D_{12} \cdot D_{34}$ *is the same as that of* $\boldsymbol{D}_{12} \cdot \boldsymbol{D}_{34}$ *for the adjoint system* (1), (5).

On the basis of Theorems I and II we classify all systems of the form (1), (2) under the following:

Type I. $D_{12} \cdot D_{34} = 0$;

Type II. $D_{12} \cdot D_{34} > 0$;

Type III. $D_{12} \cdot D_{34} < 0$.

We proceed to state a negative result.

THEOREM III. *The system* (1), (2) *of Type I satisfying conditions I–III, for which*

$$D_{13}^2 + D_{14}^2 + D_{23}^2 + D_{24}^2 = 0,$$

has no characteristic number.

Proof: We see from (11) and (12) that a system (1), (2) of Type I, for which $D_{13} = D_{14} = D_{23} = D_{24} = 0$, can be transformed into either $u(a) = 0$, $u_x(a) = 0$ or $u(b) = 0$, $u_x(b) = 0$. From the properties of the solutions of equation (1) we know that if u and u_x vanish at the same point, u vanishes identically. Hence the system possesses no characteristic number.

We turn our attention to Type I where $D_{12} \cdot D_{34} = 0$ and $D_{13}^2 + D_{14}^2 + D_{23}^2 + D_{24}^2 \neq 0$. If, as in *Case* 1 ($a$), $D_{12} = 0$ and $D_{34} = 0$, we have seen from (10) that the system (1), (2) is Sturmian. For this system there exists an infinite set of real characteristic numbers,*

$$\Lambda_1 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \Lambda_2.$$

Consider now the *Case* 1 (b), $D_{12} \neq 0$, $D_{34} = 0$, and $D_{24} \neq 0$. Then the coefficients of (2) may be transformed by $\delta = D_{24}$ into

$$\begin{pmatrix} D_{14} & D_{24} & 0 & 0 \\ D_{12} & 0 & D_{32} & D_{42} \end{pmatrix}. \tag{16}$$

It will be convenient to adopt the following notation. Let

$$L[u(x, \lambda)] = \alpha(x)u(x, \lambda) - \beta(x)K(x, \lambda)u_x(x, \lambda), \tag{17}$$

where

$$\alpha(a) = D_{14}, \qquad \alpha(b) = D_{23},$$

$$\beta(a) = D_{24}, \qquad \beta(b) = D_{24}.$$

Then conditions (2) reduce to

$$\begin{aligned} L[u(a, \lambda)] &= 0, \\ L[u(b, \lambda)] &= D_{21}u(a, \lambda). \end{aligned} \tag{18}$$

Let $u_1(x, \lambda)$ be the solution of (1) satisfying

$$u_1(a, \lambda) = D_{24}, \qquad K(a, \lambda)u_{1x}(a, \lambda) = D_{14}.$$

We may assume without loss of generality that D_{14} is positive or else $D_{14} = 0$. The characteristic equation is

$$L[u_1(b, \lambda)] = D_{21} \cdot D_{24}.$$

Let $f(\lambda) = L[u_1(b, \lambda)] + D_{12} \cdot D_{24}$. Now $f(\lambda_0) = D_{12} \cdot D_{24}$ where λ_0 is the first characteristic number of the system consisting of equation (1) together with the boundary conditions,

$$L[u(a, \lambda)] = 0,$$

$$L[u(b, \lambda)] = 0.$$

Moreover

$$\begin{aligned} \lim_{\lambda \to \Lambda_1} f(\lambda) &= \lim_{\lambda \to \Lambda_1} [D_{23}u_1(b, \lambda) - D_{24}K(b, \lambda)u_x(b, \lambda) + D_{12} \cdot D_{24}] \\ &= \lim_{\lambda \to \Lambda_1} u_1(b, \lambda) \left[D_{23} - D_{24}\frac{K(b, \lambda)u_{1x}(b, \lambda)}{u_1(b, \lambda)} + \frac{D_{12} \cdot D_{24}}{u_1(b, \lambda)} \right]. \end{aligned}$$

---

* Sturm, Jour. de Math., vol. 1, p. 100 ff.

Now

$$\lim_{\lambda \to \Lambda_1} u_1(x, \lambda) = \infty,$$

$$\lim_{\lambda \to \Lambda_1} \frac{K(x, \lambda) u_{1x}(x, \lambda)}{u_1(x, \lambda)} = \infty,$$

for $a < x \leqq b$. Hence for λ sufficiently close to Λ_1, $f(\lambda)$ is negative, or $f(\Lambda_1 + \epsilon) < 0$ if D_{12} be chosen positive and if D_{24} is positive. But $f(\lambda_0)$ is positive. Hence $f(l) = 0$ where $\Lambda_1 < l < \lambda_0$.

If $D_{14} \neq 0$, then we shall transform the coefficients of (2) by $\delta = D_{14}$ into

$$(19) \qquad \begin{pmatrix} D_{14} & D_{24} & 0 & 0 \\ 0 & D_{12} & D_{13} & D_{14} \end{pmatrix}$$

or, in terms of the notation (17),

$$L[u(a, \lambda)] = 0,$$

$$L[u(b, \lambda)] = - D_{12} K(a, \lambda) u_x(a, \lambda),$$

where

$$\alpha(a) = D_{14}, \qquad \alpha(b) = D_{13},$$

$$\beta(a) = D_{24}, \qquad \beta(b) = D_{14}.$$

We shall determine $u_1(x, \lambda)$ as the solution of (1) which satisfies

$$u(a, \lambda) = D_{24}, \qquad K(a, \lambda) u_{1x}(a, \lambda) = D_{14},$$

and the characteristic equation becomes

$$L[u_1(b, \lambda)] = - D_{12} \cdot D_{14}.$$

Let

$$f(\lambda) = D_{13} u_1(b, \lambda) - D_{14} K(b, \lambda) u_{1x}(b, \lambda) + D_{12} \cdot D_{14}.$$

Again we have $f(\lambda_0) = D_{12} \cdot D_{14}$, and the sign of $f(\Lambda_1 + \epsilon)$ is that of $- D_{14}$. If $D_{24} \geqq 0$ and D_{12} is chosen positive, if furthermore $D_{14} > 0$, then $f(\lambda_0) > 0$ and $f(\Lambda_1 + \epsilon) < 0$, or $f(l) = 0$ where $\Lambda_1 < l < \lambda_0$.

If $D_{24} = 0$ and $D_{23} \neq 0$, then we shall transform the coefficients of (2) by $\delta = D_{23}$ into

$$(20) \qquad \begin{pmatrix} D_{13} & D_{23} & 0 & 0 \\ D_{21} & 0 & D_{23} & 0 \end{pmatrix}$$

or, in terms of the notation (17),

$$L[u(a, \lambda)] = 0,$$

$$L[u(b, \lambda)] = D_{21} u(a, \lambda),$$

where

$$\alpha(a) = D_{13}, \qquad \alpha(b) = D_{23},$$

$$\beta(a) = D_{23}, \qquad \beta(b) = 0.$$

We choose $u_1(x, \lambda)$ the solution of (1) satisfying

$$u_1(a, \lambda) = D_{23}, \qquad K(a, \lambda)u_{1x}(a, \lambda) = D_{13},$$

and the characteristic equation becomes

$$D_{23}u_1(b, \lambda) + D_{12} \cdot D_{23} = 0.$$

Let $f(\lambda) = u_1(b, \lambda) + D_{12}$. If $D_{31} \geqq 0$ and D_{12} is chosen positive and if $D_{32} > 0$, then $f(\lambda_0) > 0$ and $f(\Lambda_1 + \epsilon) < 0$. Hence $f(l) = 0$ where $\lambda_0 < l < \Lambda_1$.

If $D_{14} = 0$ and $D_{13} \neq 0$, then we shall transform the coefficients of (2) by $\delta = D_{13}$ into

$$\begin{pmatrix} D_{13} & D_{23} & 0 & 0 \\ 0 & D_{12} & D_{13} & 0 \end{pmatrix} \tag{21}$$

or, in terms of the notation (17),

$$L[u(a, \lambda)] = 0,$$

$$L[u(b, \lambda)] = D_{21}K(a, \lambda)u_x(a, \lambda),$$

where

$$\alpha(a) = D_{13}, \qquad \alpha(b) = D_{13},$$

$$\beta(a) = D_{23}, \qquad \beta(b) = 0.$$

We determine $u_1(x, \lambda)$ as the solution of (1) satisfying

$$u_1(a, \lambda) = D_{23}, \qquad K(a, \lambda)u_{1x}(a, \lambda) = D_{13},$$

and the characteristic equation becomes

$$D_{13}u_1(b, \lambda) = D_{21} \cdot D_{13}.$$

Let $f(\lambda) = u_1(b, \lambda) + D_{12}$. If $D_{32} \leqq 0$ and if D_{12} is chosen positive and if $D_{31} > 0$, then $f(\lambda_0) > 0$ and $f(\Lambda_1 + \epsilon) < 0$. Hence $f(l) = 0$ where $\lambda_0 < l < \Lambda_1$.

We may dispose at once of *Case* 1 (*c*), $D_{12} = 0$, $D_{34} \neq 0$, by noting that according to Theorem I the various possibilities of *Case* 1 (*c*) are included in the adjoint system of *Case* 1 (*b*). It is a well-known theorem concerning a system (1), (2) and its adjoint (1), (5) that if $\lambda = \bar{\lambda}$ is a characteristic number of the one system, it is also a characteristic number of the other system and the index is the same for both.* In this manner the results for *Case* 1 (*b*) are carried over for *Case* 1 (*c*) with no change.

Consider now *Case* 2, $D_{12} \cdot D_{34} \neq 0$. Let

$$L_i[u(x, \lambda)] = \alpha_i(x)u(x, \lambda) - \beta_i(x)K(x, \lambda)u_x(x, \lambda) \qquad (i = 1, 2)$$

where

---

* Birkhoff, Trans. Amer. Math. Soc., vol. 9 (1908), p. 373 ff.

$$\alpha_i(a) = A_{i1}, \qquad \beta_i(a) = A_{i2},$$
$$\alpha_i(b) = A_{i3}, \qquad \beta_i(b) = A_{i4}.$$

Then (2) becomes

$$L_i[u(a, \lambda)] = L_i[u(b, \lambda)] \qquad (i = 1, 2). \tag{22}$$

If $D_{12} = D_{34}$, the system is self-adjoint. For such a system (1), (2) satisfying I–V, the writer has proved in an earlier paper* there exists an infinite set of characteristic numbers such that

$$\Lambda_1 < l_0 \leqq l_1 \leqq l_2 \leqq \cdots < \Lambda_2.$$

If $D_{12} \neq D_{34}$, we transform (2) by $\delta = D_{34}$ to

$$\begin{pmatrix} D_{13} & D_{23} & 0 & D_{43} \\ D_{14} & D_{24} & D_{34} & 0 \end{pmatrix}. \tag{23}$$

We take $D_{34} = 1$ and replace conditions (23) by

$$\begin{pmatrix} \sqrt{D_{12}} \cdot \frac{D_{31}}{\sqrt{D_{12}}} & \sqrt{D_{12}} \cdot \frac{D_{32}}{\sqrt{D_{12}}} & 0 & 1 \\ \sqrt{D_{12}} \cdot \frac{D_{14}}{\sqrt{D_{12}}} & \sqrt{D_{12}} \cdot \frac{D_{24}}{\sqrt{D_{12}}} & 1 & 0 \end{pmatrix} \tag{24}$$

for Type II and

$$\begin{pmatrix} \sqrt{D_{21}} \cdot \frac{D_{31}}{\sqrt{D_{21}}} & \sqrt{D_{21}} \cdot \frac{D_{32}}{\sqrt{D_{21}}} & 0 & 1 \\ \sqrt{D_{21}} \cdot \frac{D_{14}}{\sqrt{D_{21}}} & \sqrt{D_{21}} \cdot \frac{D_{24}}{\sqrt{D_{21}}} & 1 & 0 \end{pmatrix} \tag{25}$$

for Type III.

Let $\nabla = \sqrt{|D_{12}|}$, the positive root, and

$$\bar{\alpha}_1(a) = \frac{D_{31}}{\nabla}, \qquad \bar{\beta}_1(a) = \frac{D_{32}}{\nabla}, \qquad \bar{\alpha}_2(a) = \frac{D_{14}}{\nabla}, \qquad \bar{\beta}_2(a) = \frac{D_{24}}{\nabla}$$
$$\bar{\alpha}_1(b) = 0, \qquad \bar{\beta}_1(b) = 1, \qquad \alpha_2(b) = 1, \qquad \bar{\beta}_2(b) = 0.$$

Then (24) and (25) may be written

$$\nabla \bar{L}_i[u(a, \lambda)] = \bar{L}_i[u(b, \lambda)] \qquad (i = 1, 2) \tag{26}$$

and the adjoint set of conditions are

$$\bar{L}_i[u(a, \lambda)] = \nabla \bar{L}_i[u(b, \lambda)] \qquad (i = 1, 2)$$

where for

$$\bar{L}_i[u(a, \lambda)] = \bar{L}_i[u(b, \lambda)] \qquad (i = 1, 2) \tag{27}$$

$\bar{D}_{12} = \bar{D}_{34} = 1$, and the system (1), (27) is self-adjoint.

---

[*] Existence Theorems for the General, Real and Self-Adjoint Linear System of the Second Order. Trans. Amer. Math. Soc., vol. 19 (1918), p. 94.

Let $u_1(x, \lambda)$ and $u_2(x, \lambda)$ be two solutions of (2) satisfying

$$\bar{L}_1[u_1(a, \lambda)] = 0, \qquad \bar{L}_2[u_1(a, \lambda)] = 1,$$
$$\bar{L}_1[u_2(a, \lambda)] = 1, \qquad \bar{L}_2[u_2(a, \lambda)] = 0.$$

Direct computation will show that

$$\bar{L}_1[u_1(b)]\bar{L}_2[u_2(b)] - \bar{L}_1[u_2(b)]\cdot\bar{L}_2[u_1(b)] = -1. \tag{28}$$

Let $u = c_1u_1 + c_2u_2$ be the solution of (1) satisfying system (1), (26). Then the characteristic equation is

$$\psi(\lambda) = \begin{vmatrix} U_1(u_1) & U_2(u_1) \\ U_1(u_2) & U_2(u_2) \end{vmatrix} = 0, \tag{29}$$

or

$$\psi(\lambda) = \begin{vmatrix} -\bar{L}_1[u_1(b, \lambda)] & \nabla - \bar{L}_2[u_1(b, \lambda)] \\ \nabla - \bar{L}_1[u_2(b, \lambda)] & -\bar{L}_2[u_2(b, \lambda)] \end{vmatrix} = 0$$

which simplifies by (28) to

$$\psi(\lambda) = -1 - \nabla^2 - \nabla\{\bar{L}_1[u_2(b, \lambda)] + \bar{L}_2[u_1(b, \lambda)]\} = 0. \tag{30}$$

Now the characteristic equation of the system (1), (27) is

$$\phi(\lambda) = \bar{L}_1[u_2(b, \lambda)] + \bar{L}_1[u_1(b, \lambda)] - 2 = 0.$$

Hence (30) becomes

$$\psi(\lambda) = \nabla\cdot\phi(\lambda) - (1 - \nabla)^2 = 0.$$

Let l_0 be the first characteristic number of (1), (27). Then $\phi(l_0) = 0$ and

$$\psi(l_0) = -(1 - \nabla)^2$$

or $\psi(l_0)$ is negative. Now for λ near Λ_1, the sign of $\psi(\lambda)$ is the same as that of $\phi(\lambda)$. If $D_{24} \neq 0$, the sign of $\phi(\Lambda_1 + \epsilon)$ is that of D_{24}; and if $D_{24} = 0$, the sign is that of D_{14}.[*] Hence if $D_{24} > 0$ or if $D_{24} = 0$, $D_{14} > 0$, $\psi(\Lambda_1 + \epsilon)$ is positive. Therefore

$$\psi(\bar{l}) = 0$$

where $\Lambda_1 < \bar{l} < l_0$. Since by (4) D_{14} and D_{24} may not both vanish, this exhausts all possibilities. We may now impose on a *non-self-adjoint* system (1), (2) the following condition:

---

[*] Cf. Birkhoff, Existence and Oscillation Theorem for a Certain Boundary Value Problem, Trans. Amer. Math. Soc., vol. 15 (1909), p. 268.

VI. *A*. $D_{12} \cdot D_{34} = 0$, and

either (*a*) $D_{24}^2 + D_{14}^2 \neq 0,\ D_{14} \geqq 0,\ D_{24} \geqq 0,$

or (*b*) $D_{24}^2 + D_{14}^2 = 0,\ D_{13}^2 + D_{23}^2 = 0,\ D_{13} \leqq 0,$

$D_{23} \leqq 0.$

B. $D_{12} \cdot D_{34} \neq 0,$

either (*a*) $D_{24} > 0,$

or (*b*) $D_{24} = 0,\ D_{14} > 0.$

We may now state the result in the following form.

EXISTENCE THEOREM. *Every non-self-adjoint system* (1), (2) *satisfying conditions I–V and either VI–A or VI–B has at least one real characteristic number.*

The writer expects to consider in a later paper the system (1), (2) in which A_{ij} are functions of λ and to determine what additional restrictions are necessary to ensure the validity of the Existence Theorem.*

The following six examples illustrate the six types of non-self-adjoint systems for which the Existence Theorem has been established. In addition to the one real characteristic number assured by the Existence Theorem it may be noted that several possibilities arise:

1. The remaining characteristic numbers are real and infinite in number.
2. There is an additional finite number of real characteristic numbers, and the remaining ones are complex and infinite in number.
3. All the remaining characteristic numbers are complex and infinite in number.
4. Every value of λ is a characteristic number.

Example 1. $(D_{12} > 0,\ D_{24} > 0,\ D_{14} = 0)$

$$u'' + \lambda u = 0,$$

$$-u'(0) = 0,$$

$$-\epsilon u(0) + u'(\pi) = 0, \qquad \epsilon > 0.$$

Let $u(x, \lambda) = \cos \sqrt{\lambda} x$. Then the characteristic equation is

$$\sin \sqrt{\lambda}\pi = -\frac{\epsilon}{\sqrt{\lambda}}.$$

For this case there is always one real characteristic number,

$$\bar{\lambda}_0 = -\frac{v_0^2}{\pi^2} < 0$$

---

* Since this paper was written, this extension has been presented to the American Mathematical Society, Dec. 31, 1919. See abstract in Bull. Amer. Math. Soc., vol. 26, pp. 267–8.

where v_0 is the root of the equation

$$\sinh v = \frac{\epsilon}{\pi}\frac{1}{v}.$$

There is an infinite number of additional complex characteristic numbers,

$$\bar{\lambda}_k = (2k\pi - iv_0)^2, \qquad \text{where } k = 1, 2, 3, \cdots \text{ and } i = \sqrt{-1}.$$

There will always be an infinite number of real characteristic numbers also, viz., the real roots of

$$\sin \sqrt{\lambda}\pi = -\frac{\epsilon}{\sqrt{\lambda}}.$$

Example 2. $(D_{12} > 0,\ D_{24} = 0,\ D_{14} > 0)$.

$$u'' + \lambda u = 0,$$

$$u(0) = 0,$$

$$-\epsilon u'(0) + u'(\pi) = 0, \qquad \epsilon > 0.$$

Let

$$u(x, \lambda) = \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}}.$$

Then the characteristic equation is

$$\cos \sqrt{\lambda}\pi = \epsilon.$$

If $\epsilon \leqq 1$,

$$\sqrt{\bar{\lambda}_0} = \frac{1}{\pi}\cos^{-1}\epsilon, \qquad 0 \leqq \cos^{-1}\epsilon < \frac{\pi}{2},$$

$$\bar{\lambda}_0 = \left(\frac{1}{\pi}\cos^{-1}\epsilon\right)^2.$$

where $\bar{\lambda}_0$ is the first characteristic number. The others are

$$\bar{\lambda} = \left(2n \pm \frac{1}{\pi}\cos^{-1}\epsilon\right)^2.$$

If $\epsilon > 1$,

$$\bar{\lambda}_0 = -\left(\frac{\cosh^{-1}\epsilon}{\pi}\right)^2$$

and the remaining characteristic numbers are

$$\bar{\lambda}_k = \left(2k \pm i\,\frac{\cosh^{-1}\epsilon}{\pi}\right), \qquad \text{where } k = 1, 2, 3, \cdots$$

and $i = \sqrt{-1}$.

Example 3. $(D_{12} > 0,\ D_{14}^2 + D_{24}^2 = 0,\ D_{23} = 0,\ D_{13} < 0)$.

$$u'' + \lambda u = 0,$$

$$u(0) = 0,$$

$$-\epsilon u'(0) + u(\pi) = 0, \qquad \epsilon > 0.$$

Let

$$u(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}}$$

and the characteristic equation is

$$\sin \sqrt{\lambda} \pi = \epsilon \cdot \sqrt{\lambda}.$$

For $\epsilon \leqq \pi$, $\bar{\lambda}_0 = z_0^2$ where z_0 is the first real solution of the equation

$$\sin \pi z = \epsilon z.$$

For ϵ sufficiently small, there may be other real characteristic numbers.

For $\epsilon > \pi$, $\bar{\lambda}_0 = -\dfrac{v_0^2}{\pi^2}$ where v_0 is the root of the equation

$$\sinh v = \frac{\epsilon}{\pi} v.$$

For ϵ sufficiently large, other complex roots exist.

Example 4. $(D_{12} > 0,\ D_{14}^2 + D_{24}^2 = 0,\ D_{13} = 0,\ D_{23} < 0)$.

$$u'' + \lambda u = 0,$$

$$\epsilon u(0) - u(\pi) = 0,$$

$$-u'(0) = 0, \qquad \epsilon > 0.$$

Let $u(x, \lambda) = \cos \sqrt{\lambda} x$, and the characteristic equation is

$$\cos \sqrt{\lambda} \pi = \epsilon.$$

The characteristic numbers are those of Example 2.

Example 5. $(D_{12} \cdot D_{34} \neq 0,\ D_{34} = 1,\ D_{24} > 0)$.

$$u'' + \lambda u = 0,$$

$$u(0) + u'(\pi) = 0,$$

$$-\epsilon u'(0) - u(\pi) = 0, \qquad \epsilon > 0.$$

Let

$$u(x, \lambda) = C_1(\lambda) \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + C_2(\lambda) \cos \sqrt{\lambda} x,$$

and the characteristic equation is

$$\sin \sqrt{\lambda}\pi = -\frac{(1+\epsilon)\sqrt{\lambda}}{1+\epsilon\lambda}.$$

This equation admits real roots.

Example 6. $(D_{12} \cdot D_{34} \neq 0,\ D_{34} = 1,\ D_{24} = 0,\ D_{14} > 0)$.

$$u'' + \lambda u = 0,$$

$$\epsilon u(0) - u(\pi) = 0,$$

$$u'(0) + u'(\pi) = 0, \qquad \epsilon > 0.$$

Let

$$u(x, \lambda) = C_1(\lambda)\frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + C_2(\lambda) \cos \sqrt{\lambda}x,$$

and the characteristic equation is

$$(1 - \epsilon)(1 + \cos \sqrt{\lambda}\pi) = 0.$$

If $\epsilon = 1$, every value of λ is a characteristic number.

If $\epsilon \neq 1$, $\lambda_k = (2k + 1)^2$ where $(k = 0, 1, 2, \cdots)$.

University of Texas,
Austin, Texas.